

VISIBILITY IN PROXIMAL DELAUNAY MESHES

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Dedicated to the Memory of Som Naimpally

ABSTRACT. This paper introduces a visibility relation v (and the strong visibility relation $\overset{\mathbb{M}}{v}$) on proximal Delaunay meshes. A main result in this paper is that the visibility relation v is equivalent to Wallman proximity. In addition, a Delaunay triangulation region endowed with the visibility relation v has a local Leader uniform topology.

1. INTRODUCTION

Delaunay triangulations, introduced by B.N Delone [Delaunay] [3], represent pieces of a continuous space. A *triangulation* is a collection of triangles, including the edges and vertices of the triangles in the collection.

A 2D *Delaunay triangulation* of a set of sites (generators) $S \subset \mathbb{R}^2$ is a triangulation of the points in S . The set of vertices (called sites) in a Delaunay triangulation define a *Delaunay mesh*. A Delaunay mesh endowed with a nonempty set of proximity relations is a *proximal Delaunay mesh*. A proximal Delaunay mesh is an example of a proximal relator space [14], which is an extension a Szász relator space [15, 16, 17].

Let $S \subset \mathbb{R}^2$ be a set of distinguished points called *sites*, $p, q \in S$, \overline{pq} straight line segment in the Euclidean plane. A site p in a line is *visible* to another site q in the same straight line segment, provided there is no other site between p and q .

Example 1.1. Visible Points.

A pair of Delaunay triangles $\triangle(pqr), \triangle(rst)$ are shown in Fig. 1. Points r, q is visible from p but in the straight line segment \overline{ps} , s is not visible from p . Similarly, points p, r are visible from q but in the straight line segment \overline{qs} , t is not visible from q . From r , points p, q, s, t are visible. ■

A straight edge connecting p and q is a *Delaunay edge* if and only if the Voronoï region of p [6, 12] and Voronoï region of q intersect along a common line segment [5, §I.1, p. 3]. For example, in Fig. 2, the intersection of Voronoï regions V_p, V_q is a triangle edge, i.e., $V_p \cap V_q = \overline{xy}$. Hence, \overline{pq} is a Delaunay edge in Fig. 2.

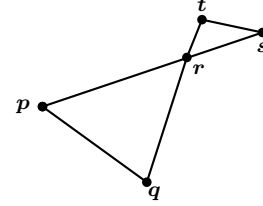
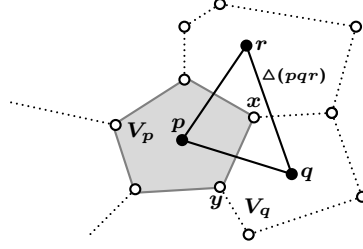


FIGURE 1. Visibility

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FIGURE 2. Delaunay triangle $\Delta(pqr)$

A triangle with vertices $p, q, r \in S$ is a *Delaunay triangle* (denoted $\Delta(pqr)$ in Fig. 2), provided the edges in the triangle are Delaunay edges. This paper introduces proximal Delaunay triangulation regions derived from the sites of Voronoï regions [12], which are named after the Ukrainian mathematician Georgy Voronoï [19].

A nonempty set A of a space X is a *convex set*, provided $\alpha A + (1 - \alpha)A \subset A$ for each $\alpha \in [0, 1]$ [1, §1.1, p. 4]. A *simple convex set* is a closed half plane (all points on or on one side of a line in \mathbb{R}^2 [6]). The edges in a Delaunay mesh are examples of convex sets. A closed set S in the Euclidean space E^n is *convex* if and only if to each point in E^n there corresponds a unique nearest point in S . For $z \in S$, a closed set in \mathbb{R}^n , $S_z = \left\{ x \in E : \|x - z\| = \inf_{y \in S} \|x - y\| \right\}$ is a *convex cone* [18].

Lemma 1.2. [6, §2.1, p. 9] *The intersection of convex sets is convex.*

Proof. Let $A, B \subset \mathbb{R}^2$ be convex sets and let $K = A \cap B$. For every pair points $x, y \in K$, the line segment \overline{xy} connecting x and y belongs to K , since this property holds for all points in A and B . Hence, K is convex. \square

2. PRELIMINARIES

Delaunay triangles are defined on a finite-dimensional normed linear space E that is topological. For simplicity, E is the Euclidean space \mathbb{R}^2 . The *closure* of $A \subset E$ (denoted $\text{cl}A$) is defined by

$$\begin{aligned} \text{cl}(A) &= \{x \in X : D(x, A) = 0\}, \text{ where} \\ D(x, A) &= \inf \{\|x - a\| : a \in A\}, \end{aligned}$$

i.e., $\text{cl}(A)$ is the set of all points x in X that are close to A ($D(x, A)$ is the Hausdorff distance [8, §22, p. 128] between x and the set A and $\|x, a\|$ is the Euclidean distance between x and a).

Let A^c denote the complement of A (all points of E not in A). The *boundary* of A (denoted $\text{bdy}A$) is the set of all points that are near A and near A^c [10, §2.7, p. 62]. An important structure is the *interior* of A (denoted $\text{int}A$), defined by $\text{int}A = \text{cl}A - \text{bdy}A$. For example, the interior of a Delaunay edge \overline{pq} are all of the points in the segment, except the endpoints p and q .

In general, a *relator* is a nonvoid family of relations \mathcal{R} on a nonempty set X . The pair (X, \mathcal{R}) is called a *relator space*. Let E be endowed with the proximal relator

$$\mathcal{R}_\delta = \left\{ \delta, \overset{\mathbb{A}}{\delta}, \underset{\mathbb{W}}{\delta}, \overset{\delta}{\delta} \right\} \text{ (Proximal Relator, cf. [14]).}$$

The Delaunay tessellated space E endowed with the proximal relator \mathcal{R}_δ (briefly, \mathcal{R}) is a *Delaunay proximal relator space*.

The proximity relations δ (near), $\overset{\delta}{\delta}$ (strongly near) and their counterparts $\underline{\delta}$ (far) and $\overset{\delta}{\underset{w}{\delta}}$ (strongly far) facilitate the description of properties of Delaunay edges, triangles, triangulations and regions. Let $A, B \subset E$. The set A is near B (denoted $A \delta B$), provided $\text{cl}A \cap \text{cl}B \neq \emptyset$ [4]. The Wallman proximity δ (named after H. Wallman [20]) satisfies the four Čech proximity axioms [2, §2.5, p. 439] and is central in near set theory [10, 11]. Sets A, B are *far* apart (denoted $A \underline{\delta} B$), provided $\text{cl}A \cap \text{cl}B = \emptyset$. For example, Delaunay edges $\overline{pq} \delta \overline{qr}$ are near, since the edges have a common point, *i.e.*, $q \in \overline{pq} \cap \overline{qr}$ (see, *e.g.*, $\overline{pq} \delta \overline{qr}$ in Fig. 2). By contrast, edges $\overline{pr}, \overline{xy}$ have no points in common in Fig. 2, *i.e.*, $\overline{pr} \underline{\delta} \overline{xy}$.

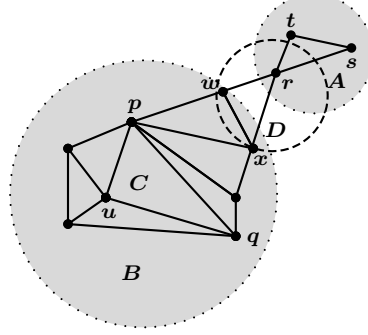


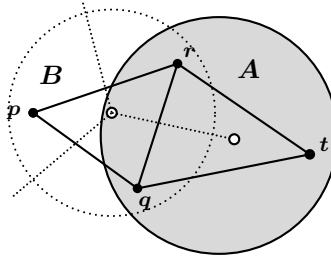
FIGURE 3. Far

Voronoi regions V_p, V_q are strongly near (denoted $V_p \overset{\delta}{\delta} V_q$) if and only if the regions have a common edge. For example, $V_p \overset{\delta}{\delta} V_q$ in Fig. 2. *Strongly near* Delaunay triangles have a common edge. Delaunay triangles $\triangle(pqr)$ and $\triangle(qrt)$ are strongly near in Fig. 4, since edge \overline{qr} is common to both triangles. In that case, we write $\triangle(pqr) \overset{\delta}{\delta} \triangle(qrt)$.

Nonempty sets $A \overset{\delta}{\underset{w}{\delta}} C$ are strongly far apart (denoted A, C), provided $C \subset \text{int}(\text{cl}B)$ and $A \underline{\delta} B$.

Example 2.1. Far and Strongly Far Sets.

In the Delaunay mesh in Fig. 3, sets A and B have no points in common. Hence, $A \underline{\delta} B$ (A is far from B). Also in Fig. 3, let $C = \{\triangle(pqu)\}$. Consequently, $C \subset \text{int}(\text{cl}B)$, such that triangle $\triangle(pqu)$ lies in the interior of the closure of B . Hence, $A \overset{\delta}{\underset{w}{\delta}} C$. ■

FIGURE 4. Strongly Visible Sets $A \overset{\delta}{\underset{w}{\delta}} B$

Let A, B be subsets in a Delaunay mesh, $\triangle(pqr) \in B, \triangle(qrt) \in A$. Subsets A, B in a Delaunay mesh are *visible* to each other (denoted $A v B$), provided at least one triangle vertex $x \in \text{cl}A \cap \text{cl}B$. A, B are *strongly visible* to each other (denoted $A \overset{\delta}{\underset{w}{\delta}} B$), provided at least one triangle edge is common to A and B .

Example 2.2. Visibility in Delaunay Meshes.

In the Delaunay mesh in Fig. 3, $A v D$, since A and D have one triangle vertex

is common, namely, vertex r . Sets B and D in Fig. 3 are strongly visible (i.e., $B \overset{\delta}{\vee} D$), since edge \overline{wx} is common to B and D . In Fig. 3, let $C = \{\triangle(pqu)\}$. Then $C \overset{\delta}{\vee} B$, since $C \subset B$. In Fig. 4, edge \overline{qr} is common to A and B . \overline{qr} is visible from $p \in B$ and from $t \in A$. Hence, $A \overset{\delta}{\vee} B$ ■

Subsets A, B in a Delaunay mesh are *invisible* to each other (denoted $A \underline{\vee} B$), provided $\text{cl}A \cap \text{cl}B = \emptyset$, i.e., A and B have no triangle vertices in common. A, B are *strongly invisible* to each other (denoted $A \overset{v}{\vee} B$), provided $C \underline{\vee} A$ for all sets of mesh triangles $C \subset B$.

Example 2.3. Invisible and Strongly Invisible Subsets in a Delaunay Mesh.

In the Delaunay mesh in Fig. 3, A and B are not visible to each other, since $\text{cl}A \cap \text{cl}B = \emptyset$, i.e., A and B have no triangle vertices in common. In Fig. 3, let $C = \{\triangle(pqu)\}$. Then $A \overset{v}{\vee} B$ (A and B are strongly invisible to each other), since $C \underline{\vee} A$ for all sets of mesh triangles $C \subset B$. ■

3. MAIN RESULTS

The Delaunay visibility relation v is equivalent to the proximity δ .

Lemma 3.1. *Let A, B be subsets in a Delaunay mesh. $A \delta B$ if and only if $A v B$.*

Proof. $A \delta B \Leftrightarrow \text{cl}A \cap \text{cl}B \neq \emptyset \Leftrightarrow A$ and B have a triangle vertex in common if and only if $A v B$. □

Theorem 3.2. *The visibility relation v is a Wallman proximity.*

Proof. Immediate from Lemma 3.1. □

Lemma 3.3. *Let A, B be subsets in a Delaunay mesh. $A \overset{\delta}{\vee} B$ if and only if AvB .*

Proof. $A \overset{\delta}{\vee} B \Leftrightarrow \overline{pq}$ for some triangle edge common to A and $B \Leftrightarrow AvB$, since \overline{pq} is visible from a vertex in A and from a vertex in B and A and B have vertices in common. □

Theorem 3.4. *The strong visibility relation $\overset{\delta}{\vee}$ is a Wallman proximity.*

Proof. Immediate from Theorem 3.2 and Lemma 3.3. □

Theorem 3.5. *Let A, B be subsets in a Delaunay mesh. Then*

1° $A \overset{v}{\vee} B$ implies $A \underline{\vee} B$.

2° $A \overset{v}{\vee} B$ if and only if $A \overset{\delta}{\vee} B$.

Proof.

1°: Given $A \overset{v}{\vee} B$, then A and B have no triangle vertices in common. Hence, $A \underline{\vee} B$.

2°: $A \overset{v}{\vee} B$ if and only if A and B have no triangles in common if and only if $A \overset{\delta}{\vee} B$. □

Theorem 3.6 is an extension of Theorem 3.1 in [13], which results from Theorem 3.2.

Theorem 3.6. *The following statements are equivalent.*

1° $\triangle(pqr)$ is a Delaunay triangle.

2° Circumcircle $\bigcirc(pqr)$ has center $u = cV_p \cap cV_q \cap cV_r$.

3° $V_p \overset{\text{ss}}{\cap} V_q \overset{\text{ss}}{\cap} V_r$.

4° $\Delta(pqr)$ is the union of convex sets.

Let P be a polygon. Two points $p, q \in P$ are *visible*, provided the line segment \overline{pq} is in $\text{int}P$ [7]. Let $p, q \in S$, L a finite set of straight line segments and let $\overline{pq} \in L$. Points p, q are visible from each other, which implies that \overline{pq} contains no point of $S - \{p, q\}$ in its interior and \overline{pq} shares no interior point with a constraining line segment in $L - \overline{pq}$. That is, $\text{int}\overline{pq} \cap S = \emptyset$ and $\overline{pq} \cap \overline{xy} = \emptyset$ for all $\overline{xy} \in L$ [5, §II, p. 32]

Theorem 3.7. *If points in $\text{int}\overline{pq}$ are visible from p, q , then $\text{int}\overline{pq} \cap S - \{p, q\} = \emptyset$ and $\overline{pq} \cap \overline{xy} = \emptyset$ for all $\overline{xy} \in L - \overline{pq}$ for all $x, y \in S - \{p, q\}$.*

Proof. Symmetric with the proof of Theorem 3.2 [13]. \square

A *Delaunay triangulation region* \mathcal{D} is a collection of Delaunay triangles such that every pair triangles in the collection is strongly near. That is, every Delaunay triangulation region is a triangulation of a finite set of sites and the triangles in each region are pairwise strongly near. *Proximal Delaunay triangulation regions* have at least one vertex in common. From Lemma 1.2 and the definition of a Delaunay triangulation region, observe

Lemma 3.8. [13] *A Delaunay triangulation region is a convex polygon.*

Theorem 3.9. [13] *Proximal Delaunay triangulation regions are convex polygons.*

A *local Leader uniform topology* [9] on a set in the plane is determined by finding those sets that are close to each given set.

Theorem 3.10. [13] *Every Delaunay triangulation region has a local Leader uniform topology (application of [9]).*

Theorem 3.11. [13] *A Delaunay triangulation region endowed with the visibility relation v has a local Leader uniform topology.*

Proof. Let \mathcal{D} be a Delaunay triangulation region. From Theorem 3.2 and Theorem 3.10, determine all subsets of \mathcal{D} that are visible from each given subset of \mathcal{D} . For each $A \subset \mathcal{D}$, this procedure determines a family of Delaunay triangles that are visible from (near) each A . By definition, this procedure induces a local Leader uniform topology on \mathcal{D} . \square

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